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Redundancy of the transitivity constraints in the linear ordering problem

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Abstract

The linear ordering problem is an NP-hard combinatorial optimization problem with a large number of applications including triangulation of input-output matrices and aggregation of individual preferences. In this paper, we deal with a standard 0-1 integer linear programming formulation of this problem and discuss the redundancy of its inequality constraints, which are referred to as the transitivity constraints. More specifically, we introduce a hierarchical classes of the transitivity constraints and give a simple sufficient condition for each class to be redundant.

Keyword: Linear ordering problem; Integer linear programming formulation; Transitivity constraints; Redundant constraints;

1 Introduction

Let $G = (V, A)$ be a simple directed complete graph with n vertices. Namely, $A = \{(u, v) : u, v \in V, u \neq v\}$. A spanning subgraph $G[T] = (V, T)$ induced by $T \subseteq A$ is called *tournament* if either $(u, v) \in T$ or $(v, u) \in T$ holds, but not both, for each distinct pair of vertices $u, v \in V$. In addition to this, if there is no directed cycle, that is, a sequence $v_1, v_2, \ldots, v_l \in V$ such that $(v_1, v_2), (v_2, v_3), \ldots, (v_l, v_1) \in T$, then we say that $G[T]$ is an *acyclic tournament*. We observe that every acyclic tournament $G[T]$ can be uniquely associated with a *linear ordering* of V, that is, a bijection $\pi : V \to \{1, 2, \ldots, n\}$ such that $(u, v) \in T \Leftrightarrow \pi(u) < \pi(v)$.

In the *linear ordering problem*, we are given arc weights $w : A \to \mathbb{R}$ and seek an acyclic tournament that maximizes the total weight. Its applications include triangulation of input output matrices [8], estimation of chronological order of ancestry items [5] and ranking of sports teams [11], to list a few.

In spite of its extensive applications, unfortunately, the linear ordering problem is known to be NP-hard (see [3, 4] for instance). Nevertheless, there are many studies on exact algorithms (see [2, 6, 10] for instance) as well as heuristic methods (see for [9] instance). These exact algorithms are exclusively important in several applications, like voting models, where "exactly" optimal solutions are needed. When developing these exact algorithms, the following integer linear programming model, which is a focus of our study, has been used and

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discussed frequently.

$$
\begin{aligned}
(P): \quad \text{maximize} \quad & \sum_{(u,v)\in A} w_{uv}x_{uv} \\
\text{subjectto} \quad & x_{uv} + x_{vu} = 1 && (u,v \in V), \\
& x_{uv} + x_{vw} + x_{wu} \le 2 && (u,v,w \in V), \\
& x_{uv} \in \{0,1\} && (u,v \in V).
\end{aligned}
$$

Here, each 0-1 variable x_{uv} takes 1 if $(u,v) \in T$ and takes 0 otherwise. The equality constraints ensure that feasible solutions are tournaments and the inequality constraints rule out directed cycles of length 3. In other words, if $(u,v) \in T$ and $(v,w) \in T$ then $(u,w) \in T$. Hence, these constraints are referred to as the *transitivity constraints*. Note that the transitivity constraints also rule out directed cycles of length $k \ge 4$ by combining with the equality constraints. The convex hull of this feasible region is referred to as the *linear ordering polytope* (see [1, 7] for instance). Although the formulation (P) is simple, the number of transitivity constraints is $O(n^3)$, which requires huge computational resources even for relatively small n.

In this paper, we introduce a hierarchical classes of the transitivity constraints and give simple sufficient conditions for each class to be redundant. Here, we say that a set of constraints *redundant* if dropping them does not change the optimal solution set of (P). Interestingly, a specified class is revealed to be always redundant without any assumption.

2 Redundant transitivity constraints

Given arc weights w, we focus on a set A^+ of arcs corresponding to "win" or "tie". More formally, $A^+ = \{(u,v) : w_{uv} \ge w_{vu}\}$. Using this definition, let us classify the set of transitivity constraints $x_{uv} + x_{vw} + x_{wu} \le 2$ into 4 classes depending on an integer value

$$l_{uvw} = |\{(u,v),(v,w),(w,u)\} \cap A^+|,$$

which ranges from 0 to 3. One could say that l_{uvw} captures how much the corresponding transitivity constraint is likely to be violated. Our first result states that the transitivity constraints with $l_{uvw} = 0$ are redundant. More specifically, letting $(RP^{[q]})$ be a relaxation problem obtained from (P) by dropping only the transitivity constraints with $l_{ijk} \le q$, we have the following.

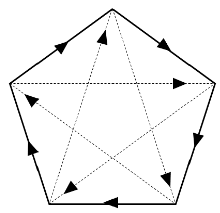
Theorem 1. *Any optimal solution of $(RP^{[0]})$ is also optimal for (P).*

If we make some assumption on the structure of (V, A^+) then we have the following.

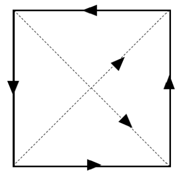
Theorem 2. *Let d be the length of a longest directed cycle in (V, A^+). If $d = 4$ then any optimal solution of $(RP^{[1]})$ is also optimal for (P). If $d = 3$ then any optimal solution of $(RP^{[2]})$ is also optimal for (P). If there is no directed cycle in (V, A^+), any optimal solution of $(RP^{[3]})$ is also optimal for (P).*

Furthermore, we observe that these sufficient conditions are tight in a sense.

Remark 1. *There is an instance $G_1 = (V_1, A_1)$ (resp. $G_2 = (V_2, A_2)$) such that $d = 5$ (resp. $d = 4$) and an optimal solution T_1 (resp. T_2) of $(RP^{[1]})$ (resp. $(RP^{[2]})$) is not feasible for (P). Figures of T_1 and T_2 are shown in Figure 2. It is easy to see that there is an instance whose optimal solution of $(RP^{[3]})$ is not feasible to (P).*



$G_1 : T_1 \subseteq A_1$ $(d = 5)$



$G_2 : T_2 \subseteq A_2$ $(d = 4)$

Figure 1: Figures of T_1 and T_2. Arcs expressed by solid (resp. dotted) lines are the members of A^+ (resp. $A \setminus A^+$) and have weights 100 (resp. 0). The other arcs which do not appear in the figures have weights 1.

3 Proofs

The following simple lemma plays an important role to show the results.

Lemma 1. *Let* $\boldsymbol{x}$ *be an optimal solution for* $(RP^{[d]})$ *for some* $d \in \{0, 1, 2, 3\}$ *and* $T_{\boldsymbol{x}} = \{(u, v) : x_{uv} = 1\}$ *be a set of arcs corresponding to* $\boldsymbol{x}$ *. If there is a directed cycle in* $(V, T_{\boldsymbol{x}})$*, then there also exists one in* $(V, T_{\boldsymbol{x}} \cap A^+)$*.*

Proof. Let s, t be a distinct pair of vertices such that there is a directed path from s to t in $(V, T_{\boldsymbol{x}})$. Here, we show that there also exists one from s to t in $(V, T_{\boldsymbol{x}} \cap A^+)$. It is easy to see that this observation completes the proof.

Suppose, to the contrary, that there is no directed path from s to t in $(V, T_{\boldsymbol{x}} \cap A^+)$. Let S be a set of vertices which are reachable from s by a directed path in $(V, T_{\boldsymbol{x}} \cap A^+)$. Furthermore let $T = V \setminus S$ and

$$F = \{(u, v) : u \in S, v \in T, x_{uv} = 1\}.$$

Then, every arc $(u, v) \in F$ must be an element of $A \setminus A^+$. Let $\boldsymbol{x}'$ be a solution obtained from $\boldsymbol{x}$ by flipping the arcs in F. Namely,

$$x'_{uv} = \begin{cases} 0 & ((u, v) \in F), \\ 1 & ((v, u) \in F), \\ x_{uv} & (\text{otherwise}). \end{cases}$$

It is not difficult to see that this operation never violate the transitivity constraints which are satisfied by $\boldsymbol{x}$. Hence, $\boldsymbol{x}'$ is a feasible solution of $(RP^{[d]})$. On the other hand, as mentioned above, we have $(u, v) \in A \setminus A^+$, that is, $w_{uv} < w_{vu}$ for $(u, v) \in F$. This means that $\boldsymbol{x}'$ attains strictly better objective value that of $\boldsymbol{x}$. This contradicts to the optimality of $\boldsymbol{x}$ in $(RP^{[d]})$. □

Proof of Theorem 1

Let $\bar{\boldsymbol{x}}$ be an optimal solution of $(RP^{[0]})$ and $T_{\bar{\boldsymbol{x}}}$ be a set of arcs corresponding to $\bar{\boldsymbol{x}}$. The goal is to show that there is no directed cycle in $(V, T_{\bar{\boldsymbol{x}}})$.

Suppose, to the contrary, that there is a directed cycle in $(V, T_{\bar{\boldsymbol{x}}})$. Then, by Lemma 1, there is a directed cycle $v_1, v_2, \ldots, v_k$ in $(V, T_{\bar{\boldsymbol{x}}} \cap A^+)$. Since $(v_1, v_2), (v_2, v_3) \in A^+$, we have $l_{v_1 v_2 v_3} \geq 2 > 0$, which means that $(RP^{[0]})$ has the transitivity constraints $x_{v_1 v_2} + x_{v_2 v_3} + x_{v_3 v_1} \leq 2$. Substituting $\bar{x}_{v_1 v_2} = \bar{x}_{v_2 v_3} = 1$ to this constraint, we have $\bar{x}_{v_3 v_1} = 0$, which implies that $\bar{x}_{v_1 v_3} = 1$ by an equality constraint $x_{v_1 v_3} + x_{v_3 v_1} = 1$. Next, let us focus on the transitivity constraints $x_{v_1v_3} + x_{v_3v_4} + x_{v_4v_1} \leq 2$. Since at least $(v_3, v_4) \in A^+$, $l_{v_1v_3v_4} \geq 1 > 0$ which means that $(RP^{[0]})$ has this transitivity constraints. Again, substituting $\bar{x}_{v_1v_3} = \bar{x}_{v_3v_4} = 1$ to this constraint, we have $\bar{x}_{v_1v_4} = 1$. Applying similar arguments along this directed cycle, we finally have $\bar{x}_{v_1v_k} = 1$, which contradicts to that $v_1, v_2, \ldots, v_k$ is a directed cycle in $(V, T_{\boldsymbol{x}})$.

Proof of Theorem 2

Since the last statement in Theorem 2 is clear, we only give proofs of the first two statements. Like the proof of Theorem 1, letting $\bar{\boldsymbol{x}}$ be an optimal solution of $(RP^{[1]})$ or $(RP^{[2]})$ and $T_{\bar{\boldsymbol{x}}} = \{(u, v) : \bar{x}_{uv} = 1\}$ be a set of arcs corresponding to this solution, we show that there is no directed cycle in $(V, T_{\boldsymbol{x}})$ under the assumptions $d = 4$ or $d = 3$. Suppose, to the contrary, that there is a directed cycle in $(V, T_{\bar{\boldsymbol{x}}})$. Then, again, by Lemma 1, there is a directed cycle $v_1, v_2, \ldots, v_k$ in $(V, T_{\bar{\boldsymbol{x}}} \cap A^+)$.

Let us consider the case when $d = 4$. In this case, the length k of the directed cycle is 3 or 4. Let $\bar{\boldsymbol{x}}$ be an optimal solution of $(RP^{[1]})$. When $k = 3$, since $l_{v_1v_2v_3} = 3 > 1$, $(RP^{[1]})$ has the transitivity constraint $x_{v_1v_2} + x_{v_2v_3} + x_{v_3v_1} \leq 2$ ruling out a directed cycle v_1, v_2, v_3 in $(V, T_{\bar{\boldsymbol{x}}})$, which yields a contradiction. When $k = 4$, since $l_{v_1v_2v_3} \geq 2 > 1$ and $l_{v_1v_3v_4} \geq 2 > 1$, $(RP^{[1]})$ has both of the transitivity constraints $x_{v_1v_2} + x_{v_2v_3} + x_{v_3v_1} \leq 2$ and $x_{v_1v_3} + x_{v_3v_4} + x_{v_4v_1} \leq 2$. Substituting $\bar{x}_{v_1v_2} = \bar{x}_{v_2v_3} = 1$ to the former constraint, we have $\bar{x}_{v_3v_1} = 0$. Then, to meat an equality constraint $x_{v_1v_3} + x_{v_3v_1} = 1$, $\bar{x}_{v_1v_3}$ must be 1. This implies that v_1, v_3, v_4 is a directed cycle in $(V, T_{\bar{\boldsymbol{x}}})$. However this is also a contradiction since the directed cycle v_1, v_3, v_4 cannot occur in $(V, T_{\bar{\boldsymbol{x}}})$ due to the transitivity constraint $x_{v_1v_3} + x_{v_3v_4} + x_{v_4v_1} \leq 2$.

Next, let us consider the case when $d = 3$. Then, the length k of the directed cycle is 3. Let $\bar{\boldsymbol{x}}$ be an optimal solution of $(RP^{[2]})$. Since $l_{v_1v_2v_3} = 3 > 2$, $(RP^{[2]})$ has the transitivity constraint $x_{v_1v_2} + x_{v_2v_3} + x_{v_3v_1} \leq 2$, which means that v_1, v_2, v_3 cannot be a directed cycle in $(V, T_{\bar{\boldsymbol{x}}})$. This is a contradiction.

4 Conclusion

In this paper, we deal with a standard integer linear programming formulation of the linear ordering problem and discuss the redundancy of its inequality constraints which are referred to as the transitivity constraints. We introduce a hierarchical classes of the transitivity constraints and give simple sufficient conditions for them to be redundant. Interestingly, a specified class is revealed to be always redundant without any assumption.

we briefly verify how often the sufficient conditions are satisfied in the real-world instances. To this end, we use the benchmark instances from a Linear Ordering library[1] (see [9] for instance). We confirmed that the length of longest cycle in these instances is equal to the number of vertices, n. Hence, for these instances, sufficient conditions in Theorem 2 are not satisfied. Nevertheless, interestingly, $(RP^{[1]})$ always gave us an optimal solution of (P) for all the instances in `Input/Output` and `SGB`. In view of this, we think that it would be an interesting future work to find a weaker sufficient condition for $(RP^{[1]})$ and (P) to share the same optimal solution set.

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