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The simplex method and the diameter of a 0-1 polytope

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Abstract

We will derive two main results related to the primal simplex method for an LP on a 0-1 polytope. One of the results is that, for any 0-1 polytope and any two vertices of it, there exists an LP instance for which the simplex method finds a path between them, whose length is at most the dimension of the polytope. This proves a well-known result that the diameter of any 0-1 polytope is bounded by its dimension. Next we show that the upper bound obtained by the authors for the number of distinct solutions generated by the simplex method is tight by constructing an LP instance on a 0-1 polytope.
Keywords: Linear programming; the number of solutions; the simplex method; 0-1 polytope.

1 Introduction

The simplex method developed by Dantzig [1] could efficiently solve a real word linear programming problem (LP), but any good upper bound for the number of iterations has not been known yet. Kitahara and Mizuno [4] extend Ye's result [8] for the Markov decision problem and obtain an upper bound for the number of distinct solutions generated by the simplex method

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with Dantzig's rule of pivoting for the standard form LP

$$\begin{array}{lll} (P_0) & \min & \boldsymbol{c}^T\boldsymbol{x}, \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

Here $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{b} \in R^m$, $\boldsymbol{c} \in R^n$, and $\boldsymbol{x} \in R^n$. The upper bound in [4] is expressed as

$$nm\frac{\gamma_P}{\delta_P}\log\left(m\frac{\gamma_P}{\delta_P}\right),$$

where δ_P and γ_P are the minimum and maximum values of all the positive elements of basic feasible solutions of (P_0). Kitahara and Mizuno [3] show that the upper bound is almost tight by using a variant of Klee-Minty's LP. Recently Kitahara and Mizuno [5] obtain a new upper bound

$$m\frac{\gamma_P\gamma'_D}{\delta_P\delta'_D} \tag{1}$$

for the number of distinct solutions generated by the primal simplex method with any rule, which chooses an entering variable whose reduced cost is negative at each iteration. Here δ'_D and γ'_D are the minimum and the maximum absolute values of all the negative elements of dual basic feasible solutions for primal feasible bases. These results are derived from the basic property of the primal simplex method that the objective function value strictly decreases whenever a solution is updated, even if the problem is degenerate. Note that the number of distinct solutions is not identical to the number of iterations when the problem is degenerate.

In this paper, we treat an LP on a 0-1 polytope $S_1 \subset R^d$, which is a convex hull of 0-1 vectors in R^d. Let $\boldsymbol{x}^s$ and $\boldsymbol{x}^t$ be any two vertices of S_1. By using the basic property of the primal simplex method, we will show that there exists an LP instance, for which the simplex method finds a path between $\boldsymbol{x}^s$ and $\boldsymbol{x}^t$ whose length is at most d. This proves a well-known result by Naddef [6] that the diameter of any 0-1 polytope in R^d is bounded by the dimension d.

We also show that the upper bound (1) is tight in the sense that there exists an LP instance on a 0-1 polytope, for which the simplex method generates exactly $m\frac{\gamma_P\gamma'_D}{\delta_P\delta'_D}$ distinct solutions.

2 The simplex method for an LP on a 0-1 polytope

In this section, we treat the following LP

$$(P_1) \quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x}, \\ \text{subject to} & \boldsymbol{x} \in S, \end{array}$$

where $S \subset R^d$ is a polytope, $\boldsymbol{c} \in R^d$ is a constant vector, and $\boldsymbol{x} \in R^d$ is a vector of variables. Let $\boldsymbol{x}^0$ be a vertex of S. For solving the problem (P_1), we use the primal simplex method from the initial vertex $\boldsymbol{x}^0$ with any rule, which chooses an entering variable whose reduced cost is negative at each iteration. The simplex method actually solves a standard form LP, which is equivalent to (P_1), and generates a sequence of its basic feasible solutions. In this section, we identify a vertex of S with a basic feasible solution of the standard form LP. The next lemma states a basic property of the simplex method, which is proved in Kitahara and Mizuno [5] for example.

Lemma 1 *The objective function value decreases whenever a basic feasible solution is updated by the primal simplex method with any rule, which chooses an entering variable whose reduced cost is negative at each iteration.*

Note that if the problem is degenerate, a basic feasible solution may not be updated when a basis is updated by the simplex method.

Let $M(P_1)$ be the maximum difference of objective function values between two vertices of S and $L(P_1)$ be the minimum positive difference of objective function values between two adjacent vertices of S. Here two vertices are called adjacent when the segment connecting them is an edge of S. Since the objective function value decreases by at least $L(P_1)$ whenever a vertex is updated from Lemma 1, the number of distinct vertices generated by the primal simplex method for solving (P_1) is bounded by

$$\frac{M(P_1)}{L(P_1)}. \tag{2}$$

Using this upper bound, we obtain the next theorem.

Theorem 1 *Let $S_1 \subset R^d$ be any 0-1 polytope and let $\mathbf{c}$ be any d-dimensional integral vector. Then the primal simplex method for solving the LP*

$$(P_2) \quad \begin{array}{ll} \min & \boldsymbol{c}^T\boldsymbol{x}, \\ \text{subject to} & \boldsymbol{x} \in S_1 \end{array}$$

starting from any vertex of S_1 generates at most C distinct vertices, where $C = \sum_{i=1}^{d} |c_i|$.

Proof: Since any vertex of S_1 is a 0-1 vector, the difference of objective function values between two vertices of S_1 is bounded by $\sum_{i=1}^{d} |c_i|$, that is,

$$M(P_2) \leq C.$$

The objective function value at any vertex is an integer, because the vector $\boldsymbol{c}$ is integral. So the minimum positive difference of objective function values between two adjacent vertices is at least one, that is,

$$L(P_2) \geq 1.$$

Hence the number of distinct vertices generated by the primal simplex method starting from any initial vertex is bounded by C from (2). ■

A finite sequence $\{\boldsymbol{x}^k | k = 0, 1, 2, \ldots, \ell\}$ of vertices of S_1 is called a path of the length ℓ on S_1, if any two consecutive vertices $\boldsymbol{x}^k$ and $\boldsymbol{x}^{k+1}$ are adjacent. A sequence of distinct vertices generated by the simplex method is a path. We obtain the next result for the length of a path between two vertices on a 0-1 polytope.

Theorem 2 *Let $S_1 \subset R^d$ be any 0-1 polytope and $\boldsymbol{x}^s$ and $\boldsymbol{x}^t$ be any two vertices of S_1. Then there exists an LP instance, for which the simplex method with Bland's rule [2] (or any anticycling rule) finds a path between $\boldsymbol{x}^s$ and $\boldsymbol{x}^t$ whose length is at most d.*

Proof: Let $\boldsymbol{x}^t = (x_1^t, x_2^t, \ldots, x_d^t)^T$. We define a vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_d)^T$ by

$$c_i = \begin{cases} -1 & \text{if } x_i^t = 1, \\ 1 & \text{if } x_i^t = 0. \end{cases}$$

Obviously $\boldsymbol{x}^t$ is the unique optimal vertex of (P_2). The simplex method starting from $\boldsymbol{x}^s$ with Bland's rule always finds the optimal vertex $\boldsymbol{x}^t$ in a finite number of iterations. Then the number of distinct vertices generated by the simplex method is at most

$$C = \sum_{i=1}^{d} |c_i| = d$$

from Theorem 1. Hence the simplex method finds a path between $\boldsymbol{x}^s$ and $\boldsymbol{x}^t$ whose length is at most d. ■

The diameter of a polytope is the maximum length of the shortest paths between its two vertices. From the theorem above, we can easily obtain a well-known result by Naddef [6] for the diameter of any 0-1 polytope.

Corollary 1 *The diameter of any 0-1 polytope in R^d is at most d.*

By using this result, Naddef [6] shows that the Hirsch conjecture is true for 0-1 polytopes, although Santos [7] recently constructs a counterexample of the conjecture for general polytopes.

3 Tightness of the upper bound by Kitahara and Mizuno

Recall that the standard form LP is expressed as

$$(P_0) \quad \begin{array}{ll} \min & \boldsymbol{c}^T \boldsymbol{x}, \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

Let $\boldsymbol{x}^0$ be a basic feasible solution of the problem (P_0). For solving the problem (P_0) from the initial basic feasible solution $\boldsymbol{x}^0$, we use the primal simplex method with any rule, which chooses an entering variable whose reduced cost is negative at each iteration. Then the objective function value decreases whenever an iterate is updated.

Kitahara and Mizuno [5] show that the maximum difference of objective function values between two basic feasible solutions of (P_0) is bounded by $m\gamma_P\gamma'_D$ and that the minimum positive difference of objective function values between two adjacent basic feasible solutions of (P_0) is at least $\delta_P\delta'_D$, that is,

$$M(P_0) \leq m\gamma_P\gamma'_D \text{ and } L(P_0) \geq \delta_P\delta'_D.$$

Hence the ratio (2) is bounded by (1) which is an upper bound derived in [5] for the number of distinct solutions generated by the primal simplex method. We show that the upper bound is tight in the next theorem.

Theorem 3 *The upper bound (1) is tight in the sense that there exists an LP instance on a 0-1 polytope for which the primal simplex method generates exactly $m\frac{\gamma_P\gamma'_D}{\delta_P\delta'_D}$ distinct solutions.*

Proof: We define an LP instance on the m-dimensional cube

$$\begin{array}{ll} \min & -\boldsymbol{e}^T \boldsymbol{x}, \\ \text{subject to} & \boldsymbol{x} \leq \boldsymbol{e}, \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

or its standard form LP

$$(P_3) \quad \begin{array}{ll} \min & -\boldsymbol{e}^T \boldsymbol{x}, \\ \text{subject to} & \boldsymbol{x} + \boldsymbol{u} = \boldsymbol{e}, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{u} \geq \boldsymbol{0}, \end{array}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)^T$ is a vector of variables, $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)^T$ is a vector of slacks, and $\boldsymbol{e} = (1, 1, \ldots, 1)^T$. The dual problem of (P_3) is

$$\begin{array}{lll} (D_3) & \max & \boldsymbol{e}^T\boldsymbol{y}, \\ & \text{subject to} & \boldsymbol{y} \leq -\boldsymbol{e}, \\ & & \boldsymbol{y} \leq \boldsymbol{0}, \end{array}$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, m)^T$ is a vector of dual variables.

It is easy to see that for any subset $K \subset \{1, 2, \ldots, m\}$ the point $(\boldsymbol{x}^K, \boldsymbol{u}^K)$ defined by

$$\begin{array}{l} x_i^K = 1,\ u_i^K = 0 \text{ for any } i \in K, \\ x_i^K = 0,\ u_i^K = 1 \text{ for any } i \notin K, \\ \boldsymbol{x}^K = (x_1^K, x_2^K, \ldots, x_m^K)^T, \\ \boldsymbol{u}^K = (u_1^K, u_2^K, \ldots, u_m^K)^T \end{array} \tag{3}$$

is a basic feasible solution of (P_3) and conversely any basic feasible solution is expressed as (3) for some $K \subset \{1, 2, \ldots, n\}$. Thus the feasible region of the problem (P_3) is a 0-1 polytope and we have that

$$\delta_P = 1 \text{ and } \gamma_P = 1,$$

where δ_P and γ_P are the minimum and maximum values of all the positive elements of basic feasible solutions of (P_3). Similarly for any subset $K \subset \{1, 2, \ldots, m\}$ the point $\boldsymbol{y}^K$ defined by

$$\begin{array}{l} y_i^K = -1 \text{ for any } i \in K, \\ y_i^K = 0 \text{ for any } i \notin K, \\ \boldsymbol{y}^K = (y_1^K, y_2^K, \ldots, y_m^K)^T \end{array}$$

is a basic solution of (D_3) and any basic solution is expressed as above. Hence

$$\delta'_D = 1 \text{ and } \gamma'_D = 1,$$

where δ'_D and γ'_D are the minimum and the maximum absolute values of all the negative elements of basic feasible solutions of (D_3) for primal feasible bases.

Note that the dual basic solution $\boldsymbol{y}^K$ is feasible only when $K = \{1, 2, \ldots, n\}$. So the optimal solution of (P_3) is

$$\boldsymbol{x}^* = (1, 1, \ldots, 1)^T,\ \boldsymbol{u}^* = (0, 0, \ldots, 0)^T.$$

Suppose that the initial solution is

$$\boldsymbol{x}^0 = (0, 0, \ldots, 0)^T,\ \boldsymbol{u}^0 = (1, 1, \ldots, 1)^T.$$

Since the feasible region of (P_3) is the m-dimensional cube, the length of the shortest path between $(\boldsymbol{x}^0, \boldsymbol{u}^0)$ and $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ is m. So the primal simplex method starting from the initial solution $(\boldsymbol{x}^0, \boldsymbol{u}^0)$ finds the optimal solution $(\boldsymbol{x}^*, \boldsymbol{u}^*)$ by generating at least m distinct solutions. On the other hand, the number of distinct solutions generated is at most $m\frac{\gamma_P \gamma'_D}{\delta_P \delta'_D}$, which is equal to m. Hence the primal simplex method generates exactly $m\frac{\gamma_P \gamma'_D}{\delta_P \delta'_D}$ distinct solutions. ■

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