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Yuichi TAKANO, Jun-ya GOTOH

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Tokyo Institute of Technology

2-12-1 Ookayama, Meguro-ku, Tokyo 152-8552, JAPAN
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Yuichi Takano [*] Jun-ya Gotoh [†]

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Abstract

We build a computational framework for determining an optimal dynamic asset allocation over multiple periods. To do this, a nonlinear control policy, which is a function of past returns of investable assets, is adopted. By employing a kernel method, the problem of selecting the best control policy from among nonlinear functions can be formulated as a convex quadratic optimization problem or a linear optimization problem. Numerical experiments show that our investment strategy improves investment performance in comparison with other strategies from previous studies.

Keywords: Multi-period portfolio optimization, Control policy, Kernel method, Conditional Value-at-Risk

1 Introduction

In this paper, we investigate an optimal policy of investing in financial assets over multiple periods. The importance of multi-period model for long-term investment has become widely-recognized (e.g., Mulvey et al. [17]). A motivation for the use of multi-period models rather than that of single-period models can be found in a fact that the rate of return of asset is not independent in its time series. For example, DeMiguel et al. [6] recently used a vector autoregressive (VAR) model to capture time-series dependence in stock returns and found that good out-of-sample performance could be achieved on the basis of the VAR model. Their results encourage us to use multi-period portfolio optimization models together with an appropriate uncertainty modeling.

[*]Department of Industrial Engineering and Management, Graduate School of Decision Science and Technology, Tokyo Institute of Technology, 2-12-1-W9-77 Ookayama, Meguro-ku, Tokyo 152-8552, JAPAN. E-mail: `takano.y.ad@m.titech.ac.jp`

[†]Department of Industrial and Systems Engineering, Chuo University, 1-13-27 Kasuga, Bunkyo-ku, Tokyo 112-8551, JAPAN. E-mail: `jgoto@indsys.chuo-u.ac.jp`

The multi-period model was first framed as a stochastic control problem [15, 16, 22] (see Infanger [10] for detailed references). Stochastic control aims to design the optimal control policy (or controller) for managing dynamical systems under uncertainty. Although optimal decision rules for portfolio selection and consumption have been established at early stage in [15, 16, 22], it is very difficult to handle stochastic control problems of practical size because of heavy numerical burden. Consequently, a number of studies have focused on stochastic programming models in which the optimal portfolios are determined instead of optimizing the control policy for managing a portfolio (see, e.g., [4, 5, 7, 12, 14, 18, 27]). Most of the stochastic programming models employ either a simulated path structure or a scenario tree structure for representing the uncertainty of asset returns (see, e.g., [8, 26]), and these models can be integrated into a hybrid (bundling simulated path) model by Hibiki [9].

The simulated path model describes multi-period scenarios of asset returns using a number of simulated paths. It can be easily calibrated to real market behavior, but the "non-anticipativity condition" is required in the optimization on the model so as to prevent investment decisions from depending on future observations on simulated paths. Without this condition, different investment decisions could be made from one scenario to another and, accordingly, could be meaningless.

By contrast, the scenario tree model enables one to make conditional investment decisions at each future state in response to the observations of the history realized until the state. It has been demonstrated in, e.g., [11, 13] that stock returns are serially dependent; therefore, it is probably effective to dynamically rebalance the portfolio on the basis of the observed realization at the time of the decision. The scenario tree model, however, is disadvantageous in that the size of the optimization problem grows exponentially as the number of time periods increases.

In view of these facts, we shall use a control policy, which is represented as a function of portfolio adjustments, in the simulated path model. Although, by using the control policy, we can make conditional investment decisions in the simulated path model as well as in the scenario tree model, the use of control policy generally leads to infinite-dimensional and nonconvex optimization.

One remedy for this drawback is to employ a "sub-optimal" solution, namely, to restrict possible decision rules to the class of control policies that are affine functions of the past outcomes (see, e.g., [1, 2, 3, 19, 25]). Although this sub-optimal solution is effective from the computational aspect, it is clear that this approach may not make the best control policy.

The purpose of this paper is to build a computational framework for determining an optimal control policy among nonlinear functions for dynamic asset allocation over multiple periods. To the best of our knowledge, no studies have ever tried to find an optimal nonlinear control policy by solving a computationally tractable optimization problem. The multi-period portfolio optimization model we consider has been devised by references to [1, 2, 3]. However, our model differs from those in that

- ▷ our model is a scenario-based stochastic programming model, and thus, we can optimize the nonlinear control policy by utilizing the kernel method;
- ▷ we employ the conditional value-at-risk (CVaR) which has desirable properties as a risk measure (see, e.g., [20, 21]), whereas [1, 2, 3] employ the variance as a risk measure.

The kernel method is an engine for dealing with high nonlinearity of statistical module in machine learning (see, e.g., [23]). By utilizing the kernel method, we can formulate the simulated path model for finding an optimal control policy among nonlinear functions as a convex quadratic optimization problem. Further, by employing an $L1$-norm regularization, we reduce the problem to a linear optimization problem. In computational experiments, we first generate scenarios of the rate of return of investable assets by using the one-period autoregressive model along the lines of [6] to take into account serial dependence in stock returns. We then compare the investment performance of our nonlinear control policy with those of other commonly-used models, i.e., a basic simulated path model and a model using linear control policies.

The rest of the paper is organized as follows: In Section 2, we describe a portfolio dynamics and present the basic simulated path model. In Section 3, we formulate the problem of optimizing a nonlinear control policy by utilizing the kernel method. Numerical results are given in Section 4, and conclusions are drawn in Section 5.

2 Portfolio Dynamics and Basic Optimization Model

In this section, following a mathematical description of a portfolio dynamics, we formulate the basic multi-period portfolio optimization model with the simulated path structure.

2.1 Preliminaries and Portfolio Dynamics

The terminology and notation used in this paper are defined as follows:

Index Sets

$\mathcal{I} := \{1, 2, ..., I\}$	:	index set of investable financial assets (where asset 1 is cash)
$\mathcal{S} := \{1, 2, ..., S\}$	:	index set of given scenarios (or simulated paths)
$\mathcal{T} := \{1, 2, ..., T\}$	:	index set of planning time periods

Decision Variables

$u_i(t)$	:	adjustment of asset i at the beginning of period t	$(i \in \mathcal{I},\ t \in \mathcal{T})$
$u_{i,s}(t)$	:	adjustment of asset i at the beginning of period t under scenario s	$(i \in \mathcal{I},\ t \in \mathcal{T} \setminus \{1\},\ s \in \mathcal{S})$
$x_{i,s}(t)$	:	investment amount in asset i at the end of period t under scenario s	$(i \in \mathcal{I},\ t \in \mathcal{T},\ s \in \mathcal{S})$
$x_i^+(0)$	:	investment amount in asset i at the beginning of the first period	$(i \in \mathcal{I})$
$x_{i,s}^+(t)$	:	investment amount in asset i at the beginning of period $t+1$ under scenario s	$(i \in \mathcal{I},\ t \in \mathcal{T} \setminus \{T\},\ s \in \mathcal{S})$
$v_s(t)$	:	portfolio value at the end of period t under scenario s	$(t \in \mathcal{T},\ s \in \mathcal{S})$

Given Constants

$\bar{x}_i(0)$	:	the initial holdings of asset i	$(i \in \mathcal{I})$
$C(t)$	:	net cash flow at the beginning of period t	$(t \in \mathcal{T})$
$R_{i,s}(t)$	:	total return of asset i in period t under scenario s	$(i \in \mathcal{I},\ t \in \mathcal{T},\ s \in \mathcal{S})$
P_s	:	occurrence probability of scenario s	$(s \in \mathcal{S})$

Figure 1 illustrates a portfolio dynamics under the scenario s. We assume that there are no transaction costs and that one has an initial portfolio $\bar{x}_i(0)$, $i \in \mathcal{I}$. If the investor has no initial endowments, $\bar{x}_i(0)$ are set to 0 for all $i \in \mathcal{I}$.

One starts investing by adjusting the portfolio as follows:

$$x_i^+(0) = \bar{x}_i(0) + u_i(1). \tag{1}$$

By definition of the total return of each asset, the investment amount changes over the first period:

$$x_{i,s}(1) = R_{i,s}(1)\, x_i^+(0). \tag{2}$$

Similarly to the first period, we rebalance the portfolio at the beginning of period $t \in \mathcal{T} \setminus \{1\}$:

$$x_{i,s}^+(t-1) = x_{i,s}(t-1) + u_{i,s}(t), \tag{3}$$

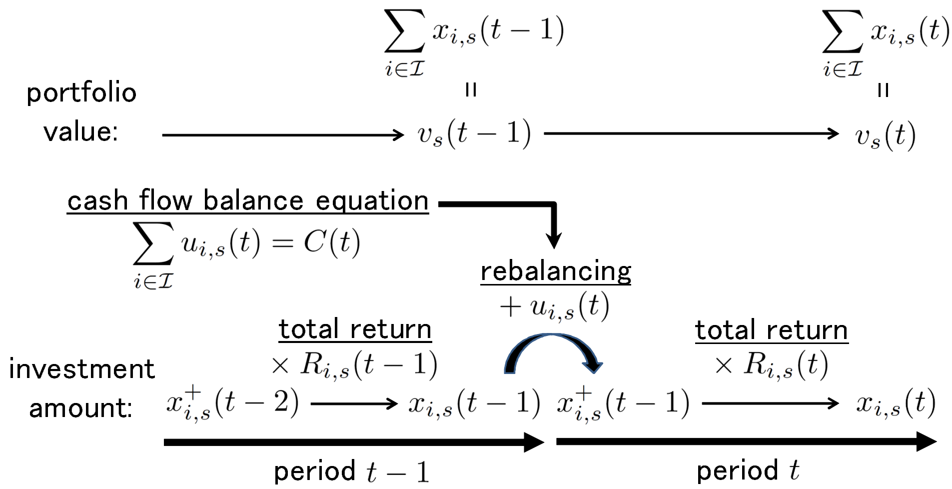

Figure 1: Portfolio Dynamics under the Scenario s

and the investment amount at the end of period $t \in \mathcal{T} \setminus \{1\}$ are as follows:

$$x_{i,s}(t) = R_{i,s}(t)\, x^+_{i,s}(t-1). \tag{4}$$

Consequently, the following portfolio dynamics equations are derived from (1) through (4):

$$\begin{aligned} x_{i,s}(1) &= R_{i,s}(1)\left(\bar{x}_i(0) + u_i(1)\right), \\ x_{i,s}(t) &= R_{i,s}(t)\left(x_{i,s}(t-1) + u_{i,s}(t)\right),\ t \in \mathcal{T} \setminus \{1\}. \end{aligned} \tag{5}$$

The portfolio value at the end of period $t \in \mathcal{T}$ under the scenario s is the sum of investments:

$$v_s(t) = \sum_{i\in\mathcal{I}} x_{i,s}(t),$$

and the expected portfolio value at the end of period $t \in \mathcal{T}$ is $\sum_{s\in\mathcal{S}} P_s v_s(t)$.

Also, the adjustments must satisfy the following cash flow balance equations in each period:

$$\sum_{i\in\mathcal{I}} u_i(1) = C(1),$$

$$\sum_{i\in\mathcal{I}} u_{i,s}(t) = C(t),\ t \in \mathcal{T} \setminus \{1\}.$$

When the self-financing portfolio is considered, the net cash flow $C(t)$ is set to 0 for all $t \in \mathcal{T}$.

2.2 Basic Optimization Model

The investment performance of portfolio selection models is usually assessed by using measures of profitability and risk. In this paper, we use the expected portfolio value as a measure of

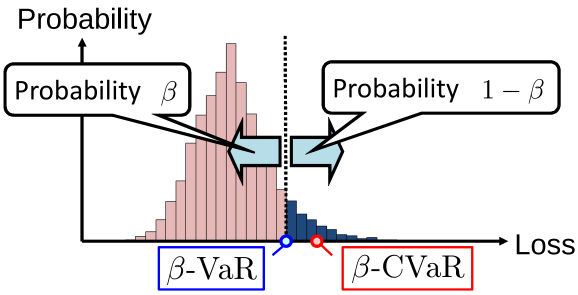

Figure 2: Value-at-Risk and Conditional Value-at-Risk

profitability and the conditional value at risk (CVaR) as a measure of risk. CVaR is known as a risk measure having desirable computational and theoretical properties (see, e.g., [20, 21] for the details).

Let $\beta \in (0,1)$ denote a confidence level. β-CVaR can then be explained as the conditional expectation of a random loss exceeding the β-value-at-risk (β-VaR), which is the β-quantile of the random loss (see Figure 2). Now the random loss is defined as the negative of the portfolio value at the end of period t, i.e., $-v_s(t)$, and the corresponding CVaR in each period is the optimal value of the following linear optimization problem (see [21]):

$$\min \left\{ a(t) + \frac{1}{1-\beta} \sum_{s \in \mathcal{S}} P_s z_s(t) \;\middle|\; z_s(t) \geq -v_s(t) - a(t),\ z_s(t) \geq 0,\ s \in \mathcal{S} \right\},$$

where $a(t)$ and $z_s(t)$ are decision variables for calculating the CVaR in period $t \in \mathcal{T}$.

In order to take into account the investment performance in all periods, the following weighted sum of the expected portfolio value is employed as a measure of profitability:

$$\sum_{t \in \mathcal{T}} \eta(t) \sum_{s \in \mathcal{S}} P_s v_s(t), \tag{6}$$

the following weighted sum of CVaR is employed as a measure of risk:

$$\sum_{t \in \mathcal{T}} \theta(t) \left(a(t) + \frac{1}{1-\beta} \sum_{s \in \mathcal{S}} P_s z_s(t) \right), \tag{7}$$

and the following weighted sum of measures of profitability and risk is employed as the objective:

$$(1-\alpha) \sum_{t \in \mathcal{T}} \theta(t) \left(a(t) + \frac{1}{1-\beta} \sum_{s \in \mathcal{S}} P_s z_s(t) \right) - \alpha \sum_{t \in \mathcal{T}} \eta(t) \sum_{s \in \mathcal{S}} P_s v_s(t),$$

where

$\eta(t)$: given nonnegative weight of the expected portfolio value at the end of period t $(t \in \mathcal{T})$

$\theta(t)$: given nonnegative weight of the CVaR at the end of period t $(t \in \mathcal{T})$

α : trade-off parameter between profitability and risk, $\alpha \in (0, 1)$

Moreover, we impose the limit constraints on the investment proportion, $x_{i,s}(t)/v_s(t)$, as follows:

$$L_i v_s(t) \leq x_{i,s}(t) \leq U_i v_s(t),$$

where L_i and U_i are lower and upper limits, respectively, of the investment proportion in asset $i \in \mathcal{I}$. For instance, L_i are set to 0 when short-sales are not allowed.

The basic multi-period portfolio optimization model is formulated as the following linear optimization problem:

$$\left|\begin{array}{lll} \text{minimize} & (1-\alpha)\sum_{t\in\mathcal{T}}\theta(t)\left(a(t)+\frac{1}{1-\beta}\sum_{s\in\mathcal{S}}P_s z_s(t)\right)-\alpha\sum_{t\in\mathcal{T}}\eta(t)\sum_{s\in\mathcal{S}}P_s v_s(t) & \cdots(8.\,\mathrm{a}) \\ \text{subject to} & z_s(t) \geq -v_s(t) - a(t),\ z_s(t) \geq 0,\ s \in \mathcal{S},\ t \in \mathcal{T} & \cdots(8.\,\mathrm{b}) \\ & x_{i,s}(1) = R_{i,s}(1)\left(\bar{x}_i(0) + u_i(1)\right),\ i \in \mathcal{I},\ s \in \mathcal{S} & \cdots(8.\,\mathrm{c}) \\ & x_{i,s}(t) = R_{i,s}(t)\left(x_{i,s}(t-1) + u_i(t)\right),\ i \in \mathcal{I},\ s \in \mathcal{S},\ t \in \mathcal{T}\setminus\{1\} & \cdots(8.\,\mathrm{d}) \\ & v_s(t) = \sum_{i\in\mathcal{I}} x_{i,s}(t),\ s \in \mathcal{S},\ t \in \mathcal{T} & \cdots(8.\,\mathrm{e}) \\ & \sum_{i\in\mathcal{I}} u_i(t) = C(t),\ t \in \mathcal{T} & \cdots(8.\,\mathrm{f}) \\ & L_i v_s(t) \leq x_{i,s}(t) \leq U_i v_s(t),\ i \in \mathcal{I},\ s \in \mathcal{S},\ t \in \mathcal{T}. & \cdots(8.\,\mathrm{g}) \end{array}\right. \quad (8)$$

It should be noted that $u_i(t)$ are used in place of $u_{i,s}(t)$ in the problem (8) in order to satisfy the non-anticipativity condition, which makes the adjustments, $u_{i,s}(t)$, independent of the future total returns, $R_{i,s}(k)$, $k \geq t$.

3 Optimization of Control Policy

3.1 Control Policy for Investment Decision

The basic optimization model (8) determines the value of adjustments $u_i(t)$ for "all" periods $t \in \mathcal{T}$ at the beginning of planning horizon. This is clearly disadvantageous because only $u_i(1)$ are here-and-now decisions and $u_i(t)$, $t \in \mathcal{T} \setminus \{1\}$, are wait-and-see decisions in the multi-stage stochastic programming model (see, e.g., [24]). That is, only $u_i(1)$ must be fixed at the beginning of planning horizon, and it is possible to determine $u_i(t),\ t \in \mathcal{T} \setminus \{1\}$, on the basis of information available at the end of period $t-1$, i.e., on the actual values of the investment amounts $x_{i,s}(k)$ and the total returns $R_{i,s}(k)$ for $1 \le k \le t-1$ (see also Figure 1). By taking advantage of such available information and exploiting the serial dependence in stock returns, we have a chance of improving the investment performance of an portfolio optimization model.

In order to make conditional investment decisions, we employ the control policy. The control policy is defined as a function of the past investment amounts and the past total returns. Specifically, with a control policy $\mathcal{F}_{i,t}$, the quantity of the adjustments $u_{i,s}(t)$ are determined as follows:

$$u_{i,s}(t) = \mathcal{F}_{i,t}\left(\boldsymbol{x}_s(t-1), \boldsymbol{R}_s(t-1)\right),\ t \in \mathcal{T} \setminus \{1\}, \tag{9}$$

where

$$\begin{aligned} \boldsymbol{x}_s(t) &:= \left(x_{i,s}(k);\ i \in \mathcal{I},\ 1 \le k \le t\right), \\ \boldsymbol{R}_s(t) &:= \left(R_{i,s}(k);\ i \in \mathcal{I},\ 1 \le k \le t\right). \end{aligned}$$

Note that the control policies $\mathcal{F}_{i,t}$ are independent of the scenario s; therefore, using the control policy does not violate the non-anticipativity condition. Additionally, the adjustments $u_{i,s}(t)$ depend on the past outcomes $\boldsymbol{x}_s(t-1)$ and $\boldsymbol{R}_s(t-1)$, and consequently, we can make conditional investment decisions with respect to each scenario.

Next, we show that the past investment amounts, $\boldsymbol{x}_s(t-1)$, can be omitted from the control policy (9) by following Calafiore and Campi [1]. To begin with, we derive the following expression by successively using the portfolio dynamics equations (5):

$$\begin{aligned} x_{i,s}(t) &= R_{i,s}(t)\left(x_{i,s}(t-1) + u_{i,s}(t)\right) \\ &= R_{i,s}(t)\, x_{i,s}(t-1) + R_{i,s}(t)\, u_{i,s}(t) \\ &= R_{i,s}(t) R_{i,s}(t-1)\left(x_{i,s}(t-2) + u_{i,s}(t-1)\right) + R_{i,s}(t)\, u_{i,s}(t) \\ &= R_{i,s}(t) R_{i,s}(t-1)\, x_{i,s}(t-2) + R_{i,s}(t) R_{i,s}(t-1)\, u_{i,s}(t-1) + R_{i,s}(t)\, u_{i,s}(t) \\ &\ \ \vdots \\ &= G_{i,s}(1,t)\left(\bar{x}_i(0) + u_i(1)\right) + \sum_{k=2}^{t} G_{i,s}(k,t)\, u_{i,s}(k), \end{aligned} \tag{10}$$

where

$$G_{i,s}(t_1, t_2) := R_{i,s}(t_1) R_{i,s}(t_1+1)\ \cdots\ R_{i,s}(t_2-1) R_{i,s}(t_2),\ t_1 \le t_2.$$

It follows from (9) and (10) that

$$x_{i,s}(t) = G_{i,s}(1,t)\left(\bar{x}_i(0) + u_i(1)\right) + \sum_{k=2}^{t} G_{i,s}(k,t)\,\mathcal{F}_{i,k}\left(\boldsymbol{x}_s(k-1), \boldsymbol{R}_s(k-1)\right).$$

We can see from this that $x_{i,s}(t)$ can be expressed by $x_{j,s}(k),\ j \in \mathcal{I},\ 1 \le k \le t-1$, and $R_{j,s}(k),\ j \in \mathcal{I},\ 1 \le k \le t$. By repeating the similar procedure for $x_{i,s}(t-1)$, $x_{i,s}(t-2)$, ..., we can see that $x_{i,s}(t)$ is expressed as a function of $R_{j,s}(k),\ j \in \mathcal{I},\ 1 \le k \le t$, of the form $\boldsymbol{x}_s(t) = \mathcal{H}_t\left(\boldsymbol{R}_s(t)\right)$. It follows that

$$u_{i,s}(t) = \mathcal{F}_{i,t}\left(\mathcal{H}_{t-1}\left(\boldsymbol{R}_s(t-1)\right), \boldsymbol{R}_s(t-1)\right),\ i \in \mathcal{I},\ s \in \mathcal{S},\ t \in \mathcal{T} \setminus \{1\}.$$

Since $u_{i,s}(t)$ can be expressed by a function only of $\boldsymbol{R}_s(t-1)$, it is clear that the following control policies have the same ability as the control policies (9) to design investment strategies:

$$u_{i,s}(t) = \mathcal{F}_{i,t}\left(\boldsymbol{R}_s(t-1)\right),\ t = \mathcal{T} \setminus \{1\}.$$

More specifically, we shall consider a control policy of the form:

$$u_{i,s}(t) = \boldsymbol{w}_i(t)^\top \boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_s(t-1)\right),\ t = \mathcal{T} \setminus \{1\}, \tag{11}$$

where $\boldsymbol{\phi}_{i,t}$ are nonlinear mappings from $\mathbb{R}^{I \times (t-1)}$ to $\mathbb{R}^{N_{i,t}}$. After the fashion of machine learning, we call the image of the mapping a feature vector. Note that $\boldsymbol{w}_i(t) \in \mathbb{R}^{N_{i,t}}$ are parameter vectors to be determined and that each element of $\boldsymbol{w}_i(t)$ represents the weight of the corresponding feature. A simple example of features would be the following polynomials of degree two:

$$\boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_s(t-1)\right) = \left(R_{i,s}(k)R_{j,s}(k)\,;\ i,j \in \mathcal{I},\ i \ge j,\ 1 \le k \le t\right),\ \text{and } N_{i,t} = \frac{I(I+1)}{2} \times t.$$

It is clear that high-dimensional feature vector enables one to ensure a greater variety of investment strategies.

3.2 Control Policy Optimization Using the Kernel Method

We could solve a problem of optimizing the control policy (11) after the feature vector functions $\boldsymbol{\phi}_{i,t}$ are defined properly. However, the problem to be solved will be intractable if high-dimensional or infinite-dimensional feature vectors are employed.

A reasonable option to overcome this difficulty is restricting the class of control policies to the linear control policy as in [1, 2, 3, 19, 25]. The following control policies are linear mappings of the past total returns of μ periods:

$$u_{i,s}(t) = \hat{u}_i(t) + \sum_{k=\max\{t-\mu,\,1\}}^{t-1} \sum_{j\in\mathcal{I}} r_{i,j}(k,t)\left(R_{j,s}(k) - \bar{R}_j(k)\right),\ i\in\mathcal{I},\ s\in\mathcal{S},\ t\in\mathcal{T}\setminus\{1\}, \quad (12)$$

where $\bar{R}_i(t) := \sum_{s\in\mathcal{S}} P_s R_{i,s}(t)$. $\hat{u}_i(t)$ represent the nominal adjustment actions, and $r_{i,j}(k,t)$ represent control actions against the past total returns. Both $\hat{u}_i(t)$ and $r_{i,j}(k,t)$ will be decision variables in the following optimization problem.

Note that the control policy (12) is not exactly the same as those used in [1, 2, 3] because their models are not scenario-based. Although the linear control policy (12) leads to a tractable linear optimization problem, nonlinear adjustment actions cannot be implemented.

Thus, in this paper, we utilize the kernel method in order to take into account highly-nonlinear adjustment actions. The kernel method is a class of algorithm for analyzing nonlinear and complex data in machine learning (see, e.g., [23]). The greatest merit of the kernel method is that it enables us to estimate an optimal function (11) without explicitly computing in the high-dimensional or infinite-dimensional feature space, which is referred to as the kernel trick in the context of machine learning.

We begin by introducing the regularization terms, $\|\boldsymbol{w}_i(t)\|^2$, i.e., the square of the Euclidean norm of $\boldsymbol{w}_i(t)$, to our problem. By suppressing the rise in the value of the regularization terms, we prevent the control policy from overfitting the total returns $R_{i,s}(t)$ used in the problem. This is a commonly-used approach for enhancing the generalization capability in kernel methods (see, e.g., [23]). By adding the regularization terms to the objective, we consider the following problem of optimizing the control policy (11):

$$
\left|\begin{array}{lll}
\text{minimize} & (1-\alpha)\displaystyle\sum_{t\in\mathcal{T}}\theta(t)\left(a(t)+\frac{1}{1-\beta}\sum_{s\in\mathcal{S}}P_s z_s(t)\right)-\alpha\sum_{t\in\mathcal{T}}\eta(t)\sum_{s\in\mathcal{S}}P_s v_s(t) & \\
 & \quad +\lambda\displaystyle\sum_{i\in\mathcal{I}\setminus\{1\}}\sum_{t\in\mathcal{T}\setminus\{1\}}\|\boldsymbol{w}_i(t)\|^2 & \cdots(13.\,\mathrm{a})\\
\text{subject to} & z_s(t)\ge -v_s(t)-a(t),\ z_s(t)\ge 0,\ s\in\mathcal{S},\ t\in\mathcal{T} & \cdots(13.\,\mathrm{b})\\
 & x_{i,s}(1)=R_{i,s}(1)\left(\bar{x}_i(0)+u_i(1)\right),\ i\in\mathcal{I},\ s\in\mathcal{S} & \cdots(13.\,\mathrm{c})\\
 & x_{i,s}(t)=R_{i,s}(t)\left(x_{i,s}(t-1)+u_{i,s}(t)\right),\ i\in\mathcal{I},\ s\in\mathcal{S},\ t\in\mathcal{T}\setminus\{1\} & \cdots(13.\,\mathrm{d})\\
 & v_s(t)=\displaystyle\sum_{i\in\mathcal{I}}x_{i,s}(t),\ s\in\mathcal{S},\ t\in\mathcal{T} & \cdots(13.\,\mathrm{e})\\
 & \displaystyle\sum_{i\in\mathcal{I}}u_i(1)=C(1);\quad \sum_{i\in\mathcal{I}}u_{i,s}(t)=C(t),\ s\in\mathcal{S},\ t\in\mathcal{T}\setminus\{1\} & \cdots(13.\,\mathrm{f})\\
 & L_i v_s(t)\le x_{i,s}(t)\le U_i v_s(t),\ i\in\mathcal{I},\ s\in\mathcal{S},\ t\in\mathcal{T}, & \cdots(13.\,\mathrm{g})\\
 & u_{i,s}(t)=\boldsymbol{w}_i(t)^\top\boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_s(t-1)\right),\ i\in\mathcal{I}\setminus\{1\},\ s\in\mathcal{S},\ t\in\mathcal{T}\setminus\{1\}, & \cdots(13.\,\mathrm{h})
\end{array}\right. \tag{13}
$$

where $\lambda > 0$ is a regularization parameter. Note from the constraints (13. h) that the control policy is not applied to cash ($i = 1$). This is because the adjustments of cash, $u_{1,s}(t)$, are uniquely determined from the adjustments of other assets through the cash flow balance equations (13. f).

Let

$$\mathcal{K}_{i,\ell,s}(t) := \boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_\ell(t-1)\right)^\top \boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_s(t-1)\right) \tag{14}$$

be the kernel function. We prove the following theorem in order to apply the kernel method to the problem (13):

Theorem 1 (Representer Theorem [23]) *The optimal adjustments $u^*_{i,s}(t)$ of the problem (13) can be expressed by*

$$u^*_{i,s}(t) = \sum_{\ell\in\mathcal{S}} e_{i,\ell}(t)\,\mathcal{K}_{i,\ell,s}(t),\ i\in\mathcal{I}\setminus\{1\},\ s\in\mathcal{S},\ t\in\mathcal{T}\setminus\{1\},$$

for some $e_{i,\ell}(t)$, $i\in\mathcal{I}\setminus\{1\}$, $\ell\in\mathcal{S}$, $t\in\mathcal{T}\setminus\{1\}$.

Proof. Let $\boldsymbol{w}^0_i(t)$ be the linear combination of feature vectors, $\boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_\ell(t-1)\right)$, as follows:

$$\boldsymbol{w}^0_i(t) := \sum_{\ell\in\mathcal{S}} e_{i,\ell}(t)\,\boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_\ell(t-1)\right).$$

Then, by selecting $\boldsymbol{\xi}_i(t)$ from orthogonal complement of the feature vectors, $\boldsymbol{w}_i(t)$ can generally be expressed as follows:

$$\boldsymbol{w}_i(t) = \boldsymbol{w}^0_i(t) + \boldsymbol{\xi}_i(t). \tag{15}$$

Since all the feature vectors are orthogonal to $\boldsymbol{\xi}_i(t)$, it follows from (13. h) and (15) that

$$u_{i,s}(t) = \boldsymbol{w}_i(t)^\top \boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_s(t-1)\right) = \boldsymbol{w}_i^0(t)^\top \boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_s(t-1)\right).$$

Therefore, the adjustments $u_{i,s}(t)$ are independent of $\boldsymbol{\xi}_i(t)$.

Furthermore, the regularization terms can be expressed as follows:

$$\lambda \sum_{i \in \mathcal{I}\setminus\{1\}} \sum_{t \in \mathcal{T}\setminus\{1\}} \|\boldsymbol{w}_i(t)\|^2 = \lambda \sum_{i \in \mathcal{I}\setminus\{1\}} \sum_{t \in \mathcal{T}\setminus\{1\}} \left(\|\boldsymbol{w}_i^0(t)\|^2 + \|\boldsymbol{\xi}_i(t)\|^2\right),$$

by the orthogonality of $\boldsymbol{w}_i^0(t)$ and $\boldsymbol{\xi}_i(t)$. The minimum of the regularization terms is therefore obtained when $\boldsymbol{\xi}_i(t) = \boldsymbol{0}$, which means that

$$\boldsymbol{w}_i(t) = \boldsymbol{w}_i^0(t) = \sum_{\ell \in \mathcal{S}} e_{i,\ell}(t)\, \boldsymbol{\phi}_{i,t}\left(\boldsymbol{R}_\ell(t-1)\right). \tag{16}$$

So this proof is completed from (13. h), (14) and (16). ■

Theorem 1 stated that the optimal adjustments, $u_{i,s}^*(t)$, can be computed without any concern for the dimensions, $N_{i,t}$, of the feature vectors.

Considering that the regularization terms in the problem (13) can also be expressed by

$$\lambda \sum_{i \in \mathcal{I}\setminus\{1\}} \sum_{t \in \mathcal{T}\setminus\{1\}} \|\boldsymbol{w}_i(t)\|^2 = \lambda \sum_{i \in \mathcal{I}\setminus\{1\}} \sum_{t \in \mathcal{T}\setminus\{1\}} \sum_{\ell \in \mathcal{S}} \sum_{s \in \mathcal{S}} e_{i,\ell}(t)\, e_{i,s}(t)\, \mathcal{K}_{i,\ell,s}(t)$$

from (14) and (16), the problem to be solved can finally be formulated as the following convex QP (quadratic optimization problem):

$$\left|\begin{array}{ll} \text{minimize} & (1-\alpha)\displaystyle\sum_{t \in \mathcal{T}} \theta(t)\left(a(t) + \frac{1}{1-\beta}\sum_{s \in \mathcal{S}} P_s z_s(t)\right) - \alpha \sum_{t \in \mathcal{T}} \eta(t) \sum_{s \in \mathcal{S}} P_s v_s(t) \\ & \quad + \lambda \displaystyle\sum_{i \in \mathcal{I}\setminus\{1\}} \sum_{t \in \mathcal{T}\setminus\{1\}} \sum_{\ell \in \mathcal{S}} \sum_{s \in \mathcal{S}} e_{i,\ell}(t)\, e_{i,s}(t)\, \mathcal{K}_{i,\ell,s}(t) \qquad \cdots (17.\,\mathrm{a}) \\ \text{subject to} & (13.\,\mathrm{b}), \ldots, (13.\,\mathrm{g}) \\ & u_{i,s}(t) = \displaystyle\sum_{\ell \in \mathcal{S}} e_{i,\ell}(t)\, \mathcal{K}_{i,\ell,s}(t),\ i \in \mathcal{I}\setminus\{1\},\ s \in \mathcal{S},\ t \in \mathcal{T}\setminus\{1\}. \ \cdots (17.\,\mathrm{b}) \end{array}\right. \tag{17}$$

Note that $e_{i,s}(t)$ are decision variables.

Although the problem (17) is tractable in the sense that convex QP is polynomial-time-solvable, it is more beneficial to reduce the problem to a linear optimization problem especially when large number of financial assets and/or scenarios must be dealt with. Thus, we shall employ an $L1$-norm regularization of the form:

$$\lambda \sum_{i\in\mathcal{I}\setminus\{1\}} \sum_{t\in\mathcal{T}\setminus\{1\}} \sum_{s\in\mathcal{S}} |e_{i,s}(t)|$$

in place of the Euclidean norm regularization term in (17). In view of the following relation:

$$|e_{i,s}(t)| = \min\{w_{i,s}(t) + y_{i,s}(t) \mid e_{i,s}(t) = w_{i,s}(t) - y_{i,s}(t),\ w_{i,s}(t) \geq 0,\ y_{i,s}(t) \geq 0\},$$

the problem (17) can be reduced to the following LP (linear optimization problem):

$$\left|\begin{array}{lll}\text{minimize} & (1-\alpha)\displaystyle\sum_{t\in\mathcal{T}} \theta(t) \left(a(t) + \frac{1}{1-\beta} \sum_{s\in\mathcal{S}} P_s z_s(t) \right) - \alpha \sum_{t\in\mathcal{T}} \eta(t) \sum_{s\in\mathcal{S}} P_s v_s(t) & \\ & \quad + \lambda \displaystyle\sum_{i\in\mathcal{I}\setminus\{1\}} \sum_{t\in\mathcal{T}\setminus\{1\}} \sum_{s\in\mathcal{S}} (w_{i,s}(t) + y_{i,s}(t)) & \cdots (18.\mathrm{a}) \\ \text{subject to} & (13.\mathrm{b}), \ldots, (13.\mathrm{g}) & \\ & w_{i,s}(t) \geq 0,\ y_{i,s}(t) \geq 0,\ i \in \mathcal{I}\setminus\{1\},\ s\in\mathcal{S},\ t\in\mathcal{T}\setminus\{1\} & \cdots (18.\mathrm{b}) \\ & u_{i,s}(t) = \displaystyle\sum_{\ell\in\mathcal{S}} (w_{i,\ell}(t) - y_{i,\ell}(t))\, \mathcal{K}_{i,\ell,s}(t),\ i \in \mathcal{I}\setminus\{1\},\ s\in\mathcal{S},\ t\in\mathcal{T}\setminus\{1\}, & \cdots (18.\mathrm{c})\end{array}\right. \qquad (18)$$

where $w_{i,s}(t)$ and $y_{i,s}(t)$ are decision variables for calculating the $L1$-norm regularization terms.

Note that Theorem 1 cannot be applied when the $L1$-norm is employed as the regularization terms. The problem (18), however, is clearly an approximation of the problem (17). We verify practical effectiveness of a solution to the problem (18) in the next section.

4 Numerical Experiments

In this section, numerical results are presented to show the investment performance of the kernel control policy, i.e., the control policy obtained by solving the problem (18). All computations were conducted on a Windows 7 personal computer with CORE i5 Processor (2.40GHz) and 4GB memory, and NUOPT (ver. 13.1.5), a mathematical programming software package developed by Mathematical System, Inc., was used to solve LPs.

We consider five financial assets (i.e., $I = 5$) over the planning horizon of five periods (i.e., $T = 5$), and we set the number of scenarios, S, to 200. The initial holdings are set as $\bar{x}_1(0) := 100$ and $\bar{x}_i(0) := 0$ for $i \in \mathcal{I}\setminus\{1\}$. The lower limit, L_i, and the upper limit, U_i, of the investment proportion are set to 0 and 0.5, respectively for all $i \in \mathcal{I}$. The net cash flow, $C(t)$, is 0 for all $t \in \mathcal{T}$. The occurrence probability, P_s, is $1/S$ for all $s \in \mathcal{S}$. The weights of the CVaR, $\theta(t)$, and those of the expected portfolio value, $\eta(t)$, are set as $\theta(T) = \eta(T) = 1$ and $\theta(t) = \eta(t) = 0$ for $t \in \mathcal{T} \setminus \{T\}$. We employ the following Gaussian kernel, which corresponds to an infinite-dimensional feature space, as a kernel function for the problem (18):

$$\mathcal{K}_{i,\ell,s}(t) = \exp\left(- \frac{\sum_{j \in \mathcal{I}} \sum_{k=1}^{t-1} \left(R_{j,\ell}(k) - R_{j,s}(k)\right)^2}{\sigma_{i,t}^2} \right),$$

where $\sigma_{i,t}$ are user-defined parameters.

We compare the efficiency of the following models and settings of parameters:

$$\left.\begin{array}{l}
\textsf{Basic}: \text{ the basic optimization model (8)} \\
\textsf{Linear(1)}: \text{ the optimization model using the linear control policy (12) with } \mu = 1 \\
\textsf{Linear(4)}: \text{ the optimization model using the linear control policy (12) with } \mu = 4 \\
\textsf{Kernel(a,a)}: \text{ our optimization model (18) with } \sigma_{i,t} = 0.1\sqrt{t-1} \text{ and } \lambda = 0.00001 \\
\textsf{Kernel(a,b)}: \text{ our optimization model (18) with } \sigma_{i,t} = 0.1\sqrt{t-1} \text{ and } \lambda = 0.001 \\
\textsf{Kernel(b,a)}: \text{ our optimization model (18) with } \sigma_{i,t} = 0.4\sqrt{t-1} \text{ and } \lambda = 0.00001 \\
\textsf{Kernel(b,b)}: \text{ our optimization model (18) with } \sigma_{i,t} = 0.4\sqrt{t-1} \text{ and } \lambda = 0.001
\end{array}\right\} \quad (19)$$

4.1 Scenario Generation

DeMiguel et al. [6] showed that the out-of-sample investment performance can be substantially improved by exploiting serial dependence in stock returns. Based on the vector autoregressive (VAR) model in [6], we have randomly generated scenarios of the total returns, $R_{i,s}(t)$, by using the following one-period autoregressive model of the rate of returns $\tilde{R}_i(t) - 1$, $i \in \mathcal{I} \setminus \{1\}$:

$$\tilde{R}_i(t) - 1 = \gamma_i + \sum_{j \in \mathcal{I} \setminus \{1\}} \delta_{i,j} \left(\tilde{R}_j(t-1) - 1 \right) + \tilde{\varepsilon}_i(t), \; i \in \mathcal{I} \setminus \{1\},$$

where γ_i are intercepts and $\delta_{i,j}$ are coefficients of the asset j's rate of return, and $\tilde{\varepsilon}_i(t)$ are random errors. Note that asset 1 is cash and $R_{1,s}(t)$ are set to 1 for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$, without loss of generality. It is assumed that $\tilde{\varepsilon}_i(t)$ are independently and identically distributed with respect to $t \in \mathcal{T}$.

We have collected monthly data of investment fund's base price from 2003 to 2010 from the Yahoo finance Japan[1]. Specifically, asset 2 is a fund investing in large-sized and value

[1] `http://finance.yahoo.co.jp`

stocks, asset 3 is a fund investing in large-sized and growth stocks, asset 4 is a fund investing in small-sized and value stocks, and asset 5 is a fund investing in small-sized and growth stocks.

The estimated values of parameters γ_i, $i \in \mathcal{I} \setminus \{1\}$, and δ_{ij}, $i, j \in \mathcal{I} \setminus \{1\}$, are

$$\begin{pmatrix} 0.0064 \\ 0.0035 \\ 0.0111 \\ 0.0176 \end{pmatrix} \text{ and } \begin{pmatrix} 0.404 & 0.074 & 0.108 & -0.273 \\ 0.338 & 0.073 & 0.089 & -0.259 \\ 0.539 & 0.022 & 0.235 & -0.427 \\ 0.388 & 0.381 & 0.152 & -0.437 \end{pmatrix},$$

respectively.

It is assumed that $\tilde{\varepsilon}_i(t)$ follow multivariate normal distribution with zero mean and the variance-covariance matrix $\mathbf{\Sigma}$. The estimated value of $\mathbf{\Sigma}$ is as follows:

$$\mathbf{\Sigma} = \begin{pmatrix} 0.0026 & 0.0023 & 0.0028 & 0.0030 \\ 0.0023 & 0.0024 & 0.0027 & 0.0030 \\ 0.0028 & 0.0027 & 0.0038 & 0.0036 \\ 0.0030 & 0.0030 & 0.0036 & 0.0048 \end{pmatrix}.$$

It is found from the above results that assets 4 and 5, which consist of small-sized stocks, are relatively high-risk and high-return (see the values of γ_i and diagonal elements of $\mathbf{\Sigma}$).

4.2 Performance Evaluation Methodology

We have generated two different sets of scenarios, i.e., training set $\{R_{i,s}(t) \mid i \in \mathcal{I},\ s \in \mathcal{S},\ t \in \mathcal{T}\}$ and testing set $\{R_{i,s}^{\text{out}}(t) \mid i \in \mathcal{I},\ s \in \mathcal{S}^{\text{out}},\ t \in \mathcal{T}\}$, where $\mathcal{S}^{\text{out}}$ is a set of scenarios of size S and $\mathcal{S}^{\text{out}} \cap \mathcal{S} = \emptyset$. The optimization problems (19) are formulated and solved by using the training set. Then let us denote by $u_i^*(t)$ the optimal solutions to the problem (8), by $\hat{u}_i^*(t)$ and $r_{i,j}^*(k,t)$ those to the problem of optimizing the linear control policy (12), and by $w_{i,s}^*(t)$ and $y_{i,s}^*(t)$ those to the problem (18).

In-sample performance is evaluated based on the training set. Out-of-sample performance is then evaluated as follows: In the case of basic optimization model (8), the performance of the adjustments $u_i^{\text{out}}(t) = u_i^*(t)$ are evaluated on the basis of the testing set. In the case of the linear control policy (12), out-of-sample performance is evaluated on the basis of the testing set by using the following adjustments:

$$u_i^{\mathrm{out}}(1) = u_i^*(1),\ i \in \mathcal{I},$$
$$u_{i,s}^{\mathrm{out}}(t) = \hat{u}_i^*(t) + \sum_{k=\max\{t-\mu,1\}}^{t-1} \sum_{j\in\mathcal{I}} r_{i,j}^*(k,t)\left(R_{j,s}^{\mathrm{out}}(k) - \bar{R}_j(k)\right),\ i \in \mathcal{I}\setminus\{1\},\ s \in \mathcal{S}^{\mathrm{out}},\ t \in \mathcal{T}\setminus\{1\},$$
$$u_{1,s}^{\mathrm{out}}(t) = C(t) - \sum_{i\in\mathcal{I}\setminus\{1\}} u_{i,s}^{\mathrm{out}}(t),\ s \in \mathcal{S}^{\mathrm{out}},\ t \in \mathcal{T}\setminus\{1\}.$$

For the model (18), we employ the following Gaussian kernel function for out-of-sample performance evaluation:

$$\mathcal{K}_{i,\ell,s}^{\mathrm{out}}(t) = \exp\left(-\frac{\sum_{j\in\mathcal{I}}\sum_{k=1}^{t-1}\left(R_{j,\ell}(k) - R_{j,s}^{\mathrm{out}}(k)\right)^2}{\sigma_{i,t}^2} \right),$$

and then we evaluate out-of-sample performance on the basis of the testing set by using the following adjustments:

$$u_i^{\mathrm{out}}(1) = u_i^*(1),\ i \in \mathcal{I},$$
$$u_{i,s}^{\mathrm{out}}(t) = \sum_{\ell\in\mathcal{S}} \left(w_{i,\ell}^*(t) - y_{i,\ell}^*(t)\right)\mathcal{K}_{i,\ell,s}^{\mathrm{out}}(t),\ i \in \mathcal{I}\setminus\{1\},\ s \in \mathcal{S}^{\mathrm{out}},\ t \in \mathcal{T}\setminus\{1\},$$
$$u_{1,s}^{\mathrm{out}}(t) = C(t) - \sum_{i\in\mathcal{I}\setminus\{1\}} u_{i,s}^{\mathrm{out}}(t),\ s \in \mathcal{S}^{\mathrm{out}},\ t \in \mathcal{T}\setminus\{1\}.$$

4.3 Efficient Frontier

Figures 3 and 4 show the efficient frontiers of solutions obtained by solving the problems (19). In Figures 3 and 4, the horizontal axis and the vertical axis are the expected portfolio value (6) and the CVaR (7), respectively. Each plot on a frontier in Figures 3 and 4 corresponds to a different value of the trade-off parameter, $\alpha \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$.

In-Sample Performance. Figure 3 shows the in-sample performance. It can be seen from Figure 3 that solutions to the basic optimization model (8) are dominated by solutions to other models that use control policies. Linear control policy with $\mu = 4$ performs better than that with $\mu = 1$. The frontiers of Linear(1) and Linear(4) are similar to those of Kernel(b,b) and Kernel(b,a), respectively. Solutions to the linear control policies are all dominated by those to Kernel(a,b). Moreover, solutions to Kernel(a,a) overwhelm all the other solutions.

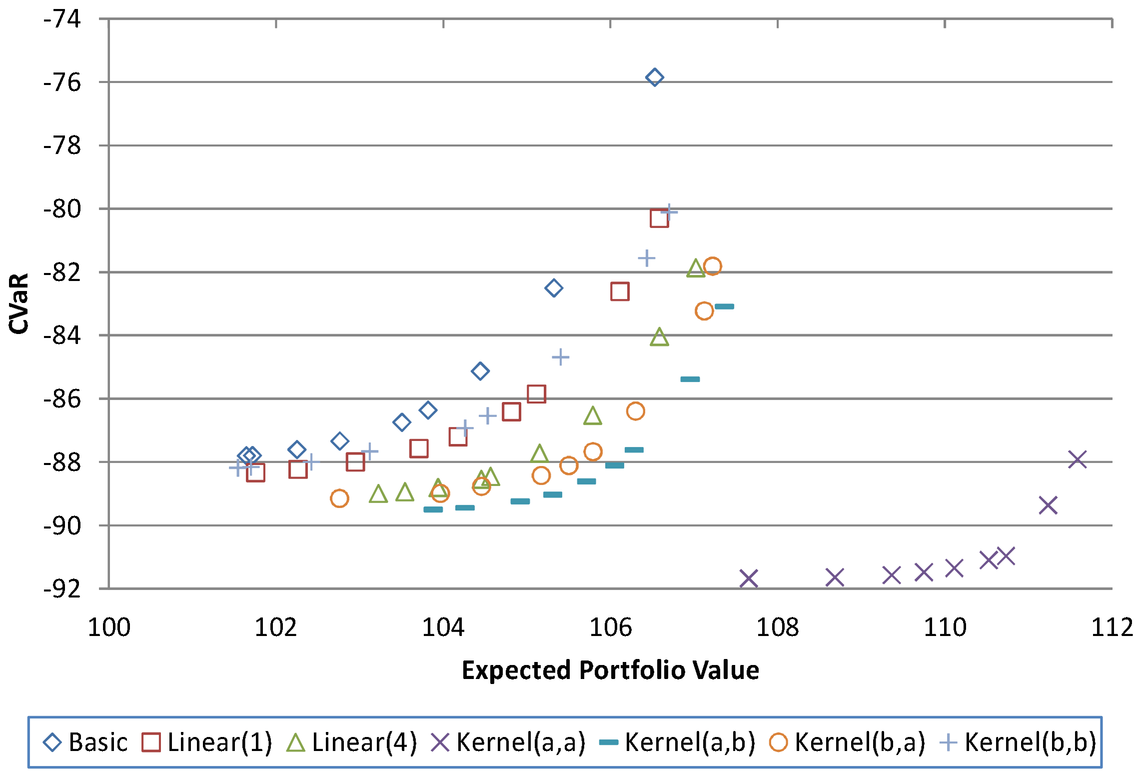

Figure 3: Efficient Frontier (in-sample, see also (19))

Out-of-Sample Performance. Figure 4 shows the out-of-sample performance. It can be found that solutions to Kernel(a,a) deteriorate. Since it performed the best in the in-sample tests, this indicates that Kernel(a,a) overfitted the training scenario set. Solutions to the basic optimization model (8) are still dominated by those to the models except Kernel(a,a); however the difference among them is smaller in the out-of-sample tests than in the in-sample tests. Solutions to Kernel(a,b) dominate those to all the other models when a high-return investment is made. Solutions to Linear(4) are dominated by those to Kernel(a,b), but the difference is not very large. Also, Linear(1) and Kernel(b,b) attain low values of the CVaR when a low-risk investment is made.

It is clear from Figures 3 and 4 that dynamic asset allocation under the nonlinear control policy is effective if we properly set parameter values. In addition, it can be noticed that the linear control policy, which is a kind of compromise solution to an intractable optimization problem, is comparable to nonlinear control policies in the out-of-sample tests.

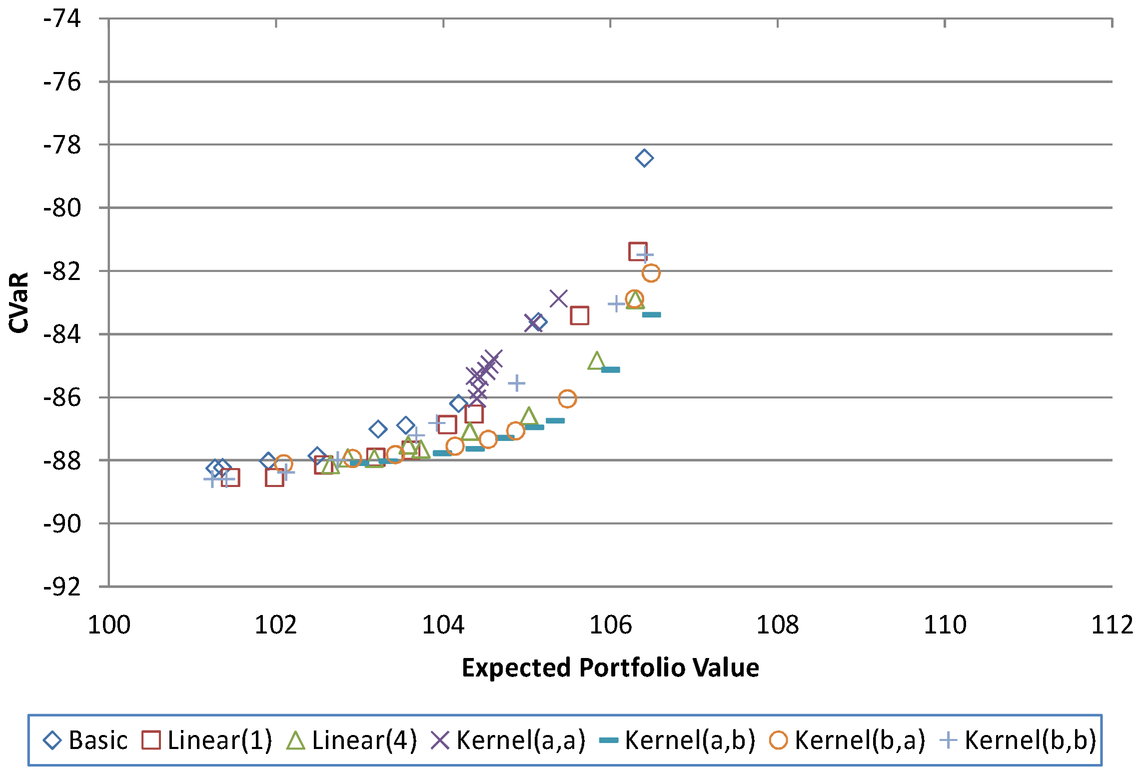

Figure 4: Efficient Frontier (out-of-sample, see also (19))

Table 1: CPU Time (in seconds, see also (19))

	Basic	Linear(1)	Linear(4)	Kernel(a,a)	Kernel(a,b)	Kernel(b,a)	Kernel(b,b)
min	2.0	2.6	5.8	163.0	140.5	154.2	149.3
average	2.3	3.0	7.0	195.9	154.8	187.5	167.0
max	2.8	3.6	8.2	250.0	181.8	213.9	195.6

Computational Time. In drawing an efficient frontier in Figure 3, we solved nine optimization problems, each corresponding to $\alpha \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$. Table 1 shows the minimum CPU time (min), the average CPU time (average) and the maximum CPU time (max) of each model. Although the investment performance is improved by our model as mentioned above, CPU time of solving our model is longer than that of solving other models.

4.4 Investment Amounts in Out-of-Sample Tests

Figure 5 shows the boxplots of the investment amounts, $x_{i,s}(t)$, in the out-of-sample tests. The boxplot displays a distribution of $\{x_{i,s}(t) \mid s \in \mathcal{S}^{\text{out}}\}$ for each asset $i \in \mathcal{I}$ and each period $t \in \mathcal{T}$. The investment amounts obtained by the basic optimization model (8) have a low dispersion

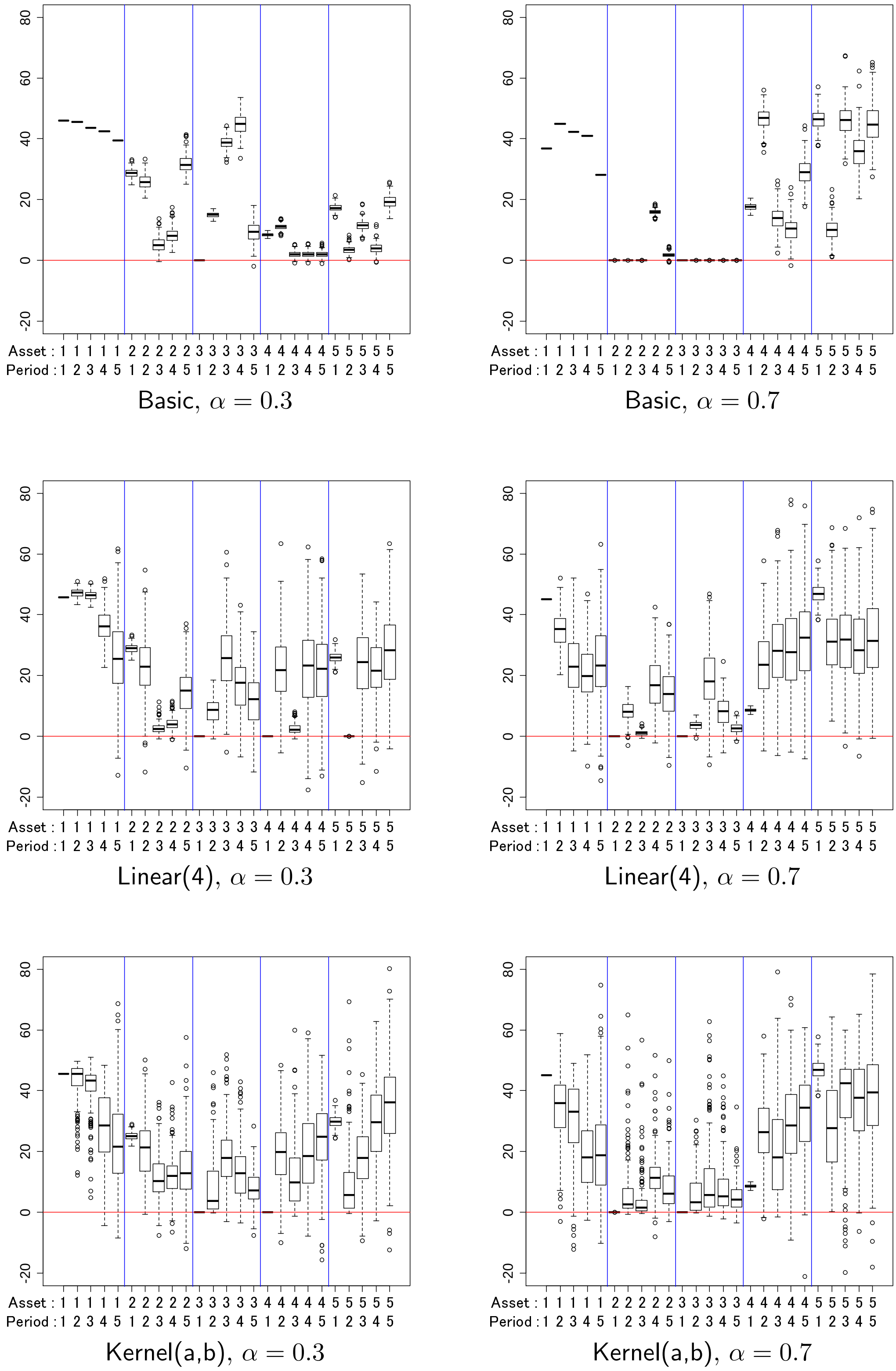

Figure 5: Investment Amounts in Out-of-Sample Tests (see also (19))

relative to those by linear and kernel control policies. We can interpret this as meaning that the control policies improve the investment performance by making flexible investment decisions with respect to each scenario. Although short-sales are not allowed by the constraints on training scenario sets, it can be seen that control policies sometimes sell short in the out-of-sample tests.

It can be noticed that when high return is preferred (i.e., $\alpha = 0.7$), Linear(4) and Kernel(a,b) make large investments in low-return assets (assets 2 and 3) compared to Basic. Similarly, when low risk is preferred (i.e., $\alpha = 0.3$), Linear(4) and Kernel(a,b) make large investments in high-risk assets (assets 4 and 5) compared to Basic. We can see from these results that the control policy achieves both high return and low risk by efficiently diversifying investments in a wide variety of assets.

5 Conclusions

We have built a computational framework to determine an optimal nonlinear control policy for dynamic asset allocation over multiple periods. By utilizing the kernel method, the problem of selecting the best control policy from among nonlinear functions has been formulated as a convex quadratic optimization problem. Further, by employing the $L1$-norm regularization, we have reduced the problem to a linear optimization problem so that we can solve it stably and efficiently by using a mathematical programming software package.

We have conducted numerical experiments to assess the investment performance of what we call the kernel control policy. Although CPU time of optimizing the kernel control policy was long compared to other models, our kernel control policy enables one to have better investment performance than the basic model and linear control policies.

A further direction of this study is to create a procedure for setting parameters of the kernel method. Our numerical experiments have shown that the investment performance is highly dependent on the parameter value; hence, how to set parameters is of practical importance.

References

[1] G.C. Calafiore and M.C. Campi, " On Two-Stage Portfolio Allocation Problems with Affine Recourse," Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference, Seville, Spain (2005).

[2] G.C. Calafiore, "Multi-Period Portfolio Optimization with Linear Control Policies," *Automatica*, Vol.44, no.10, pp.2463–2473 (2008).

[3] G.C. Calafiore, "An Affine Control Method for Optimal Dynamic Asset Allocation with Transaction Costs," *SIAM Journal on Control and Optimization*, Vol.48, no.4, pp.2254–2274 (2009).

[4] D.R. Cariño, T. Kent, D.H. Myers, C. Stacy, M. Sylvanus, A.L. Turner, K. Watanabe and W.T. Ziemba, "The Russell-Yasuda Kasai Model: An Asset/Liability Model for a Japanese Insurance Company using Multistage Stochastic Programming," *Interfaces*, Vol.24, No.1 pp.29–49 (1994).

[5] G.B. Dantzig and G. Infanger, "Multi-Stage Stochastic Linear Programs for Portfolio Optimization," *Annals of Operations Research*, Vol.45, No.1 pp.59–76 (1993).

[6] V. DeMiguel, F.J. Nogales and R. Uppal, "Stock Return Serial Dependence and Out-of-Sample Portfolio Performance," AFA 2011 Denver Meetings Paper (2010).

[7] S.-E. Fleten, K. Høyland and S.W. Wallace, "The Performance of Stochastic Dynamic and Fixed Mix Portfolio Models," *European Journal of Operational Research*, Vol.140, No.1, pp.37–49 (2002)

[8] N. Hibiki, "Multi-Period Stochastic Programming Models using Simulated Paths for Strategic Asset Allocation," *Journal of the Operations Research Society of Japan*, Vol.44, No.2, pp.169–193 (2001) (in Japanese).

[9] N. Hibiki, "A Hybrid Simulation/Tree Stochastic Optimization Model for Dynamic Asset Allocation," In B. Scherer (Eds.), *Asset and Liability Management Tools: A Handbook for Best Practice* (pp. 269–294), Risk Books, London (2003).

[10] G. Infanger, "Dynamic Asset Allocation Strategies Using a Stochastic Dynamic Programming Approach," In S.A. Zenios & W.T. Ziemba (Eds.), *Handbook of Asset and Liability Management* (pp. 199–251), Elsevier, Amsterdam (2006).

[11] N. Jegadeesh and S. Titman, "Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency," *Journal of Finance*, Vol.48, No.1, pp.65–91 (1993).

[12] M.I. Kusy and W.T. Ziemba, "A Bank Asset and Liability Management Model," *Operations Research*, Vol.34, No.3 pp.356–376 (1986).

[13] A. Lo and A.C. MacKinlay, "When Are Contrarian Profits Due to Stock Market Overreaction?," *The Review of Financial Studies*, Vol.3, No.2, pp.175–205 (1990).

[14] C.D. Maranas, I.P. Androulakis, C.A. Floudas, A.J. Berger and J.M. Mulvey, "Solving Long-Term Financial Planning Problems via Global Optimization," *Journal of Economic Dynamics & Control*, Vol.21, No.8–9, pp.1405–1425 (1997).

[15] R.C. Merton, "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics*, Vol.51, No.3, pp.247–257 (1969).

[16] R.C. Merton, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, Vol.3, No.4, pp.373–413 (1971).

[17] J.M. Mulvey, W.R. Pauling and R.E. Madey, "Advantages of Multiperiod Portfolio Models," *The Journal of Portfolio Management*, Vol.29, No.2, pp.35–45 (2003).

[18] J.M. Mulvey and H. Vladimirou, "Stochastic Network Optimization Models for Investment Planning," *Annals of Operations Research*, Vol.20, No.1, pp.187–217 (1989).

[19] J.A. Primbs and C.H. Sung, "A Stochastic Receding Horizon Control Approach to Constrained Index Tracking," *Asia-Pacific Financial Markets*, Vol.15, No.1, pp.3–24 (2008).

[20] G.Ch. Pflug, "Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk," In S. Uryasev (Eds.), *Probabilistic Constrained Optimization: Methodology and applications* (pp. 272–281), Kluwer (2000).

[21] R.T. Rockafellar and S. Uryasev, "Conditional Value-at-Risk for General Loss Distributions," *Journal of Banking & Finance*, Vol.26, No.7, pp.1443–1471 (2002).

[22] P.A. Samuelson, "Lifetime Portfolio Selection by Dynamic Stochastic Programming," *Review of Economics and Statistics*, Vol.51, No.3, pp.239–246 (1969).

[23] B. Schölkopf and A.J. Smola, *Learning with Kernels: Support Vector Machines, Regularization, Optimization, and Beyond*, MIT Press (2001).

[24] A. Shapiro, D. Dentcheva and A. Ruszczynski, *Lectures on Stochastic Programming: Modeling and Theory*, SIAM, Philadelphia (2009).

[25] J. Skaf and S. Boyd, "Multi-Period Portfolio Optimization with Constraints and Transaction Costs," Working Paper, Stanford University (2008).

[26] M.C. Steinbach, "Markowitz Revisited: Mean-Variance Models in Financial Portfolio Analysis," *SIAM Review*, Vol.43, No.1, pp.31–85 (2001).

[27] Y. Takano and J. Gotoh, "Constant Rebalanced Portfolio Optimization under Nonlinear Transaction Costs," *Asia-Pacific Financial Markets*, Vol.18, No.2, pp.191–211 (2011).